

On the asymptotic stability of a rational multi-parameter first order difference equation

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Abstract

In this part we study the dynamics of the following rational multi-parameter first order difference equation

$$x_{n+1} = \frac{ax_n^3 + bx_n^2 + cx_n + d}{x_n^3}, \quad x_0 \in \mathbb{R}^+$$

where the parameters a, b, d together with the initial condition x_0 are positive while the parameter c could accept some negative values. We investigate the equilibria and 2-cycles of this equation and analyze qualitative and asymptotic behavior of it's solutions such as convergence to an equilibrium or to a 2-cycle.

Keywords: Difference equation; equilibrium; 2-cycle; invariant interval; convergence

1 Introduction

Difference equations and discrete dynamical systems appear as both the discrete analogs of differential and delay differential equations in which time and space are not continuous and as direct mathematical models of diverse phenomena such as biology (see [2, 3]), medical sciences (see [15]), economics (see [12, 14]), military sciences (see [10, 16]), and so forth.

One of the most practical classes of nonlinear difference equations (for nonlinear difference equations see [1, 17, 18, 19])) are rational difference equations that are simply the ratio of two polynomials. Most of the work about rational type difference equations treat the case when the numerator and denominator are both linear polynomials (see the monograph of Kulenovic and Ladas [13]). Also there are some works about rational

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difference equations of order two with quadratic terms in numerator or denominator (see [4, 5, 6, 7, 8, 11]).

In this paper we analyze the global behavior of the following rational difference equations

$$x_{n+1} = \frac{ax_n^3 + bx_n^2 + cx_n + d}{x_n^3}, \quad (1.1)$$

where the parameters a, b, d together with the initial value x_0 are positive while the parameter c could accept some negative values. We study the equilibria and 2-cycles of Eq.(1.1) and show that in most cases every positive solution of this equation converges to either an equilibrium or a 2-cycle.

Suppose that the sequence $\{x_n\}$ is a solution of the first order difference equation $x_{n+1} = f(x_n)$, where $f : I \rightarrow I$, $I \subseteq \mathbb{R}$. The solution \bar{x} of the equation $x = f(x)$ is called an equilibrium (or fixed) point of this equation. The equilibrium point \bar{x} is said to be stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that for any initial condition $x_0 \in I$ with $|x_0 - \bar{x}| < \delta$, the iterates x_n satisfy $|x_n - \bar{x}| < \epsilon$ for all $n \in \mathbb{N}$. \bar{x} is said to be attracting if for all $x_0 \in I$, $\lim_{n \rightarrow \infty} x_n = \bar{x}$. \bar{x} is asymptotically stable if it is both stable and attractor. The point $p \in I$ is called a period two solution if $f^2(p) = p$, where f^2 denotes the second iterate of f . If moreover, $f(p) \neq p$ then p is called a prime period two solution. In this case the point $(p, f(p))$ is called a 2-cycle of f . An interval $I \subseteq \mathbb{R}$ is called invariant under f if $f(I) \subseteq I$. For higher order difference equations similar notions could be defined (see [9]).

2 The parameter c and equilibria

Theorem 2.1. Let $\phi(x) = \frac{ax^3 + bx^2 + cx + d}{x^3}$, $x > 0$ which is the right hand side of Eq.(1.1).

(a) The following cubic polynomial has a unique negative zero

$$Q(x) = (4a)x^3 - (b^2)x^2 - (18abd)x + 27a^2d^2 + 4db^3.$$

(b) Assume that c_- is the unique negative zero of the cubic polynomial in (a). If $c > c_-$ then nonpositive iterations of Eq.(1.1) never occur.

(c) Let $c^* = -\sqrt{3bd}$. Then $c_- < c^*$.

(d) If $c \geq c^*$ then ϕ is decreasing and has a unique equilibrium. If $c_- < c < c^*$ then ϕ has a minimum point x_m and a maximum point x_M where

$$0 < x_m = \frac{-c - \sqrt{c^2 - 3bd}}{b} < x_M = \frac{-c + \sqrt{c^2 - 3bd}}{b}.$$

Proof. (a) The facts that $Q(0) > 0$, $Q'(0) < 0$, $Q(x) \rightarrow -\infty$ as $x \rightarrow -\infty$ together with the fact that Q is a cubic polynomial prove (a).

(b) Define $F(x) = ax^3 + bx^2 + cx + d, x > 0$ which simply is the numerator of ϕ . If $F(x) > 0$ then Eq.(1.1) could not accept nonpositive iterations. Note that $F(x) > 0$ if $c \geq 0$. Now, we want to determine the negative values of c such that $F(x) > 0$. Thus, assume that $c < 0$. Since F decreases as the parameter c decreases then it's evident that for a special negative value of c there will exist a point x^* such that

$$F(x^*) = F'(x^*) = 0.$$

Some calculations show that the equation $F'(x^*) = 0$ implies that $bx^{*2} = -\frac{2b^2}{3a}x^* - \frac{bc}{3a}$ and $ax^{*3} = \frac{4b^2}{9a}x^* - \frac{2bc}{3a}$. Replace bx^{*2} and ax^{*3} into the equation $F(x^*) = 0$ to obtain

$$x^* = \frac{bc - 9ad}{6ac - 2b^2}.$$

Note that $x^* > 0$ since $c < 0$. Now, in order to obtain the value of the parameter c which corresponds to x^* replace x^* in the equation $F'(x^*) = 0$ and solve for c to obtain

$$(4a)c^3 - (b^2)c^2 - (18abd)c + 27a^2d^2 + 4db^3 = 0,$$

which simply is the equation $Q(c) = 0$ and by (a) this equation has a unique negative zero, namely c_- . Therefore, in order to avoid nonpositive iterations for Eq.(1.1) the parameter c should be greater than c_- .

(c) Note that $Q(c^*) = 6abd\sqrt{3bd} + db^3 + 27a^2d^2 > 0$, $c^* < 0$, and c_- is the unique negative zero of Q . These facts prove (c).

(d) The equation $\phi'(x) = 0$ is equivalent to the equation $bx^2 + 2cx + 3d = 0$ with the determinant $\Delta = 4(c^2 - 3bd)$. Thus if $c^* \leq c \leq -c^*$, then ϕ is decreasing on $(0, \infty)$. Also for $c > -c^*$ ϕ has no positive extremum since the summation and the product of the extremum points are negative and positive respectively. Therefore, if $c \geq -c^*$ then again ϕ is decreasing on $(0, \infty)$. So, ϕ is decreasing on $(0, \infty)$ if $c \geq c^*$. Hence, ϕ has a unique equilibrium when $c \geq c^*$. Now, assume that $c_- < c < c^*$. In this case ϕ has a minimum point $x_m = (-c - \sqrt{c^2 - 3bd})/b$ and a maximum point $x_M = (-c + \sqrt{c^2 - 3bd})/b$, both positive. The proof is complete.

In order to avoid nonpositive iterations we assume that $c > c_-$, hereafter. The next theorem deals with the number of equilibria when $c \in (c_-, c^*)$.

Theorem 2.2. Suppose that $c \in (c_-, c^*)$ and let $P(t) = t^4 - at^3 - bt^2 - ct - d$, $t > 0$. Suppose also that c_b is the unique negative root of the following quadratic polynomial

$$H(x) = 108x^2 + (108ab + 27a^3)x - 9a^2b^2 - 32b^3.$$

(a) If $c = c_b$ then P' touches the horizontal axis at $t^* = -\frac{6c_b + ab}{3a^2 + 8b} > 0$.

(b) If $c_b \geq \frac{b^2 - 12d}{3a}$ then ϕ has a unique positive equilibrium for all $c \in (c_-, c^*)$.

(c) If $c_b < \frac{b^2-12d}{3a}$ then by increasing the parameter c from c_b P touches the horizontal axis at a minimum point t_m (at first) and a maximum point t_M (next) with $0 < t_M < t_m$. Assume that c_m and c_M are the values of parameter c which correspond to t_m and t_M respectively. Then

$$c_b < c_m < c_M < c^*, \quad c_- < c_M.$$

In this case there are two subcases as follow:

- (c₁) $c_- \leq c_m$: If $c \in (c_m, c_M)$ then ϕ has three positive equilibria. For $c = c_m$, ϕ has two positive equilibria. One of them is t_m with the fact that ϕ is tangent to the 45 degree line at t_m and t_m is greater than the other equilibrium. For $c = c_M$, ϕ has again two positive equilibria. One of them is t_M with the fact that ϕ is tangent to the 45 degree line at t_M and t_M is lower than the other equilibrium. Finally, if $c \in (c_-, c_m) \cup (c_M, c^*)$ then ϕ has a unique positive equilibrium.
- (c₂) $c_m < c_-$; If $c \in (c_-, c_M)$ then ϕ has three positive equilibria. For $c = c_M$ ϕ has two positive equilibria. One of them is t_M with the fact that ϕ is tangent to the 45 degree line at t_M and t_M is lower than the other equilibrium. Finally, if $c \in (c_M, c^*)$ then ϕ has a unique positive equilibrium.

Proof. (a) With an analysis precisely similar to what was applied in Theorem 1(b) we obtain that P' touches the horizontal axis at t^* when $c = c_b$. It remains to show that $t^* > 0$. A little algebra shows that

$$H(-ab/6) = -24a^2b^2 - 9/2a^4b - 32b^3 < 0,$$

this together with the fact that $H(c_b) = 0$ imply that $c_b < -ab/6$, hence $t^* > 0$.

(b) At first note that the roots of p are simply the fixed points of ϕ . Note also that $P''(t^*) = P'(t^*) = 0$, $\partial P'(t)/\partial c = -1 < 0$ and P' is decreasing on $(0, t^*)$ and increasing on (t^*, ∞) for any c . Also, $P'(0) = -c > 0$ for all $c < 0$. These facts imply that P' has no positive root for all $c < c_b$ and two positive roots for all $c \in (c_b, 0)$. Hence P has no positive extremum for all $c \leq c_b$ and two positive extremums for all $c \in (c_b, 0)$. Therefore, if $c \leq c_b$ then P is increasing on $(0, \infty)$ and will have a unique positive root (note that by the intermediate value theorem P has at least one positive root). Now, assume that $c = c_b$. By the relation $c_b^2 = 1/108(9a^2b^2 + 32b^3 - (108ab + 27a^3)c_b)$ and by some algebra one can write

$$P(t^*) = -1/4ac_b + 1/12b^2 - d.$$

so $P(t^*) \leq 0$ since $c_b \geq (b^2 - 12d)/3a$. This fact together with the fact that $\partial P(t)/\partial c = -t < 0$ imply that P has a unique positive root when $c > c_b$, too. Thus P has a unique positive root for all $c < 0$, hence for $c \in (c_-, c^*)$. So ϕ has a unique positive equilibrium for $c \in (c_-, c^*)$.

(c) Since P' is decreasing on $(0, t^*)$ and increasing on (t^*, ∞) then the local maximum of p is lower than it's local minimum for all $c \in (c_b, 0)$. Therefore, according to the facts

that $\partial P(t)/\partial c = -t < 0$ and for $c = c_b$ $P(t^*) > 0$ (since $c_b < (b^2 - 12d)/3a$) it is evident that as the parameter c increases from c_b , P will touch the horizontal axis two times, at first in a local minimum point (namely t_m when $c = c_m$) and then in a local maximum point (namely t_M when $c = c_M$). Thus $c_b < c_m < c_M$.

Now, suppose that $c < c_m$. Then P has a unique root and by the intermediate value theorem this root is less than the maximum point of P . After that, suppose that $c = c_m$. Then P touches the horizontal axis at t_m . It is easy to show that this is equivalent to this fact that ϕ is tangent to the 45 degree line at t_m . In this case P has another root which by the intermediate value theorem is less than the maximum point of P and therefore is less than t_m . Next, assume that $c \in (c_m, c_M)$. Then, using intermediate value theorem, P has a unique root less than the maximum point of P , a unique root between the maximum and minimum points of P , and finally a unique root greater than the minimum point of P . So P has three roots in this case. Similar to the case $c = c_m$, in the case $c = c_M$, P has two roots with t_M as the lower root. Finally, similar to the case $c < c_m$, in the case $c > c_M$ P has a unique root greater than the minimum point of P .

It's easy to show that $\phi(x_m) \geq x_m > 0$ for $c \geq c_M$, where x_m is the local minimum of ϕ . Also, by Theorem 1(b) $\phi(x_m)(c - c_-) > 0$ for $c \neq c_-$ and $\phi(x_m) = 0$ for $c = c_-$. These facts together with the fact that $\partial\phi/\partial c > 0$ simply prove that $c_- < c_M$. The other details of the proof of (c₁) and (c₂) are straightforward and will be omitted. The proof is complete.

Theorem 2.3. (a) *If ϕ has a unique positive equilibrium \bar{t} then*

$$(\phi(t) - t)(t - \bar{t}) < 0, \quad t > 0, \quad t \neq \bar{t}.$$

(b) *If ϕ has two equilibria $\bar{t}_1 < \bar{t}_2$ then for $c = c_M$*

$$(\phi(t) - t)(t - \bar{t}_2) < 0, \quad t > 0, \quad t \neq \bar{t}_1, \bar{t}_2,$$

and for $c = c_m$

$$(\phi(t) - t)(t - \bar{t}_1) < 0, \quad t > 0, \quad t \neq \bar{t}_1, \bar{t}_2,$$

where c_m and c_M have introduced in Theorem 2.

(c) *If ϕ has three equilibria $\bar{t}_1 < \bar{t}_2 < \bar{t}_3$ then*

$$(\phi(t) - t)(t - \bar{t}_1)(t - \bar{t}_2)(t - \bar{t}_3) < 0, \quad t > 0, \quad t \neq \bar{t}_1, \bar{t}_2, \bar{t}_3.$$

Proof. The proof is clear (using Theorem 2) and will be omitted.

3 Two cycles and convergence

Lemma 3.1. *Assume that p is a prime period two solution of ϕ . Then p is a zero of the following sixth order polynomial*

$$G(x) = a^3x^6 + (2a^2b - ac - d)x^5 + (ab^2 - ad - bc + 2a^2c)x^4 + (2a^2d + 2abc - c^2 - bd)x^3 + (ac^2 + 2abd - 2cd)x^2 + (2acd - d^2)x + ad^2,$$

Therefore ϕ has at most three 2-cycles.

Proof. Some algebra shows that

$$\phi^2(t) - t = \frac{t^3}{(at^3 + bt^2 + ct + d)^3} (\phi(t) - t)G(t). \quad (3.1)$$

Since p is a prime period two solution then $\phi(p) \neq p$, $\phi^2(p) = p$. Hence Eq.(3.1) implies that $G(p) = 0$. Now note that if (p, q) be a 2-cycle then both p and q are roots of G . This simply shows that the number of 2-cycles of ϕ is at most three. The proof is complete.

Lemma 3.2. *Assume that $c \in (c_-, c^*)$ and \bar{t} is an equilibrium of ϕ with $x_M \leq \bar{t}$. Define*

$$G_1 = -dx_M^4 + (ad - bc)x_M^3 - (c^2 + bd)x_M^2 - 2cdx_M - d^2,$$

and

$$G_2 = a^2x_M^6 + (2ab - c)x_M^5 + (b^2 - 2d + 2ac)x_M^4 + (2ad + 2bc)x_M^3 + (c^2 + 2bd)x_M^2 + 2cdx_M + d^2.$$

(a) *Both G_1 and G_2 are positive.*

(b) $\phi^2(x_M) > x_M$.

(c) $-1 < \phi'(\bar{t}) \leq 0$.

(d) $\phi^2(t) > t$ for all $t \in [x_M, \bar{t}]$.

Proof. (a) We use the following equation in our proof frequently

$$bx_M^2 + 2cx_M + 3d = 0, \quad (3.2)$$

which is clear by the fact that x_M is an extremum of ϕ . Now, we show that G_1 is positive. Eq.(3.2) implies that

$$\begin{aligned} G_1 &= -dx_M^4 + adx_M^3 - cx_M(bx_M^2 + cx_M) - d(bx_M^2 + 2cx_M + d) \\ &= -dx_M^4 + adx_M^3 + c^2x_M^2 + 3cdx_M + 2d^2. \end{aligned} \quad (3.3)$$

Now consider the polynomial P in Theorem 2. Recall that the roots of P are the equilibria of ϕ . Since $x_M \leq \bar{t}$ then $P(x_M) < 0$, i.e., $x_M^4 < ax_M^3 + bx_M^2 + cx_M + d$. This fact together with (3.3) imply that

$$G_1 > (c^2 - bd)x_M^2 + 2cdx_M + d^2, \quad (3.4)$$

The fact that $c < c^*$ together with (3.2) and (3.4) yield

$$G_1 > 2bdx_M^2 + 2cdx_M + d^2 = d(2bx_M^2 + 2cx_M + d) = d(-2cx_M - 5d), \quad (3.5)$$

Since $c < c^*$ then the inequality $-c\sqrt{c^2 - 3bd} > -c^2 + 3bd$ holds. Some algebra show that this inequality is equivalent to the following inequality

$$-cx_M > 3d, \quad (3.6)$$

So $-2cx_M - 5d > 0$. Thus, by (3.6) $G_1 > 0$.

Next, we show that G_2 is positive. The inequality $c^2 > 3bd$, (3.2), and (3.6) imply that

$$\begin{aligned} G_2 &= a^2x_M^6 + (ab - c)x_M^5 - 2dx_M^4 + c^2x_M^2 + ax_M^3(bx_M^2 + 2cx_M + 2d) + \\ &\quad bx_M^2(bx_M^2 + 2cx_M + d) + d(bx_M^2 + 2cx_M + d) \\ &= a^2x_M^6 + (ab - c)x_M^5 - 2dx_M^4 - adx_M^3 + (c^2 - 2bd)x_M^2 - 2d^2 \\ &> a^2x_M^6 + (-cx_M - 2d)x_M^4 + a(bx_M^2 - d)x_M^3 + d(bx_M^2 - 2d) \\ &= a^2x_M^6 + (-cx_M - 2d)x_M^4 + a(-2cx_M - 4d)x_M^3 + d(-2cx_M - 5d) \\ &> a^2x_M^6 + dx_M^4 + 2adx_M^3 + d^2 \\ &> 0. \end{aligned}$$

(b) Since $x_M \leq \bar{t}$ then by Theorem 3 we obtain that $\phi(x_M) \geq x_M$. Also, it's easy to verify that

$$G(x_M) = aG_2 + x_MG_1,$$

where G is as in Lemma 3.2. So by (a) we have $G(x_M) > 0$. Therefore, the right hand side of (3.1) (for $t = x_M$) is positive. Hence, $\phi^2(x_M) > x_M$.

(c) It's evident that $\phi'(\bar{t}) \leq 0$. It remains to verify that $\phi'(\bar{t}) > -1$. Some calculations show that this inequality is equivalent to the inequality $a\bar{t}^3 - c\bar{t} - 2d > 0$, which is true by (3.6) and the fact that $\bar{t} \geq x_M$.

(d) Assume, for the sake of contradiction, that (d) is not true. By Lemma 1, ϕ has at most three 2-cycles. It's evident that this fact together with (b) and (c) imply that one and only one of the following cases is possible:

- (i) either ϕ has two prime period two solutions in the interval $[x_M, \bar{t})$ or,
- (ii) ϕ has a unique prime period two solution in the interval $[x_M, \bar{t})$ so that ϕ^2 is tangent to the 45-degree line at this point.

Assume that p is the greatest prime period two solution of ϕ in $[x_M, \bar{t})$. Then it's evident that $(\phi^2)'(p) > 1$ if (i) holds and $(\phi^2)'(p) = 1$ if (ii) holds. Therefore, in both cases we have

$$(\phi^2)'(p) \geq 1, \quad (3.7)$$

Consider the 2-cycle (p, q) . So by Eq.(1.1) we have

$$qp^3 = ap^3 + bp^2 + cp + d, \quad pq^3 = aq^3 + bq^2 + cq + d. \quad (3.8)$$

On the other hand, since $p, q > x_M$ then we obtain from (3.6) that $ap^3 - cp - 2d > 0$ and $aq^3 - cq - 2d > 0$. Therefore

$$\frac{bp^2 + 2cp + 3d}{ap^3 + bp^2 + cp + d} < 1, \quad \frac{bq^2 + 2cq + 3d}{aq^3 + bq^2 + cq + d} < 1, \quad (3.9)$$

By (3.8), (3.9), and the chain rule of calculus one can write

$$\begin{aligned} (\phi^2)'(p) = \phi'(p)\phi'(q) &= \frac{bp^2 + 2cp + 3d}{p^4} \cdot \frac{bq^2 + 2cq + 3d}{q^4} \\ &= \frac{bp^2 + 2cp + 3d}{qp^3} \cdot \frac{bq^2 + 2cq + 3d}{pq^3} \\ &= \frac{bp^2 + 2cp + 3d}{ap^3 + bp^2 + cp + d} \cdot \frac{bq^2 + 2cq + 3d}{aq^3 + bq^2 + cq + d} \\ &< 1, \end{aligned}$$

which simply contradicts (3.7). The proof is complete.

Lemma 3.3. Assume that $c \in (c_-, c^*)$ and let $c_1^* = -2\sqrt{bd}$.

(a) $c_- < c_1^*$.

(b) $\phi(x_m) > a$ if and only if $c > c_1^*$.

(c) Suppose that $c > c_1^*$. Then there exists a unique number $\eta > x_M$ such that $\phi(\eta) = \phi(x_m)$ where

$$\eta = \frac{-dx_m}{cx_m + 2d}.$$

Proof. (a) Consider the cubic polynomial Q in Theorem 1. Some computations show that $Q(c_1^*) = 4abd\sqrt{bd} + 27a^2d^2 > 0$. This fact together with the fact that c_- is the unique negative root of Q prove (a).

(b) The inequality $\phi(x_m) > a$ is equivalent to the inequality $bx_m^2 + cx_m + d > 0$ which by the equation $bx_m^2 + 2cx_m + 3d = 0$ is equivalent to $-cx_m - 2d > 0$. Some algebra show that the later inequality is equivalent to $c\sqrt{c^2 - 3bd} > -c^2 + 2bd$. Note that both sides of this inequality are negative since $c < c^*$. By squaring both sides of this inequality and simplifying it we obtain $c^2 < 4bd$ or, equivalently $c > c_1^*$.

(c) By (b) we have $\phi(x_m) > a$. Also we know that $\phi(x_M) > \phi(x_m)$. Therefore, by the intermediate value theorem and the fact that ϕ is decreasing on (x_M, ∞) there should exists a unique number $\eta > x_M$ such that $\phi(\eta) = \phi(x_m)$. Simplify the equation $\phi(\eta) = \phi(x_m)$ to obtain

$$(bx_m^2 + cx_m + d)\eta^2 + (cx_m^2 + dx_m)\eta + dx_m = 0,$$

which using the equation $bx_m^2 + 2cx_m + 3d = 0$ is equivalent to the following equation

$$(-cx_m - 2d)\eta^2 + (cx_m^2 + dx_m)\eta + dx_m = 0,$$

some algebra show that the later equation equals the following equation

$$(x_m - \eta)((cx_m + 2d)\eta + dx_m) = 0,$$

Therefore since $\eta > x_m$ we have $\eta = -dx_m/(cx_m + 2d)$. The proof is complete.

Lemma 3.4. *Assume that (p, q) is a 2-cycle of ϕ with $p < q$ and c_m and c_M are as in Theorem 2.*

(a) *If ϕ has a unique equilibrium \bar{t} then*

$$p < \bar{t} < q.$$

(b) *If ϕ has two equilibria $\bar{t}_1 < \bar{t}_2$ then for $c = c_m$ one of the following cases is possible*

$$p < \bar{t}_1 < q < \bar{t}_2, \quad \text{or} \quad p < \bar{t}_1, \quad q > \bar{t}_2,$$

and for $c = c_M$ the following case holds

$$p < \bar{t}_1, \quad q > \bar{t}_2.$$

(c) *If ϕ has three equilibria $\bar{t}_1 < \bar{t}_2 < \bar{t}_3$ then one of the following cases is possible*

$$p < \bar{t}_1 < q < \bar{t}_2, \quad \text{or} \quad p < \bar{t}_1, \quad q > \bar{t}_3.$$

Proof. (a) If $p < q < \bar{t}$ then by Theorem 3(a), $p = \phi(q) > q$, a contradiction. A similar contradiction obtains when $\bar{t} < p < q$. Lemma 3.2(d) plays an essential role for the rest of theorem whose proofs are somehow easy and similar and will be omitted.

The following theorem is about the convergence of solutions of Eq.(1.1) when $c \in [c^*, \infty)$.

Theorem 3.1. *Assume that $c \geq c^*$, \bar{t} is the unique equilibrium of ϕ , and $\{t_n\}_{n=0}^\infty$ is a positive solution for Eq.(1.1). Consider polynomial G in Lemma 1 and suppose that G has no iterated root of order even. Then there are four cases to consider as follow:*

(a) *ϕ has no 2-cycle; In this case $\{t_n\}$ converges to \bar{t} .*

(b) *ϕ has one 2-cycle (p, q) with $p < \bar{t} < q$; In this case $\{t_n\}$ converges to the 2-cycle (p, q) if $t_0 \neq \bar{t}$. Otherwise, $\{t_n\}$ simply converges to \bar{t} .*

(c) *ϕ has two 2-cycles (p_1, q_1) , (p_2, q_2) with $p_1 < p_2 < \bar{t} < q_2 < q_1$; In this case $\{t_n\}$ converges to \bar{t} if $t_0 \in (p_2, q_2)$ and converges to the 2-cycle (p_2, q_2) if $t_0 = p_2$ or q_2 . Otherwise, $\{t_n\}$ converges to the 2-cycle (p_1, q_1) .*

(d) *ϕ has three 2-cycles (p_1, q_1) , (p_2, q_2) , (p_3, q_3) with $p_1 < p_2 < p_3 < \bar{t} < q_3 < q_2 < q_1$; In this case $\{t_n\}$ converges to the 2-cycle (p_3, q_3) if $t_0 \in (p_2, q_2) \setminus \{\bar{t}\}$, converges to \bar{t} if $t_0 = \bar{t}$, and converges to the 2-cycle (p_2, q_2) if $t_0 = p_2$ or q_2 . Otherwise, $\{t_n\}$ converges to the 2-cycle (p_1, q_1) .*

Proof. ϕ^2 is increasing on $(0, \infty)$ since ϕ is decreasing on $(0, \infty)$. This is a key point in this theorem and facilitates the proof. The proof is somehow similar for all cases. So we only give the proof for one of them. Let's prove (b). By Lemma 3.4(a), $p < \bar{t} < q$. Also, $\phi^2(t) \rightarrow a$ as $t \rightarrow 0^+$ and $\phi^2(t) \rightarrow \phi(a)$ as $t \rightarrow \infty$. On the other hand, since G has no iterated root of order even then ϕ^2 is not tangent to the 45 degree line at p and q . Thus

$$(\phi^2(t) - t)(t - p)(t - \bar{t})(t - q) < 0.$$

The proof of convergence is easy using the above inequality and the fact that ϕ^2 is increasing. The proof is complete.

Remark 3.1. In Theorem 3.1, it is assumed (for the sake of simplicity) that G has no iterated root of order even. This assumption is not necessary. In fact if (p, q) is a 2-cycle for ϕ in which p and q are iterated roots of G of order even then ϕ^2 is tangent to the 45 degree line at p and q . In this case ϕ^2 is semiasymptotic stable (attracting from one side and repelling from the other side) at points p and q . There is a similar theorem for such a case and there's no need to mention it.

On the other hand, all cases in Theorem 3.1 are possible. In all of the following examples cases (a)-(d) in Theorem 4 occur respectively:

- (i) If $a = b = c = d = 1$ then ϕ has no 2-cycle.
- (ii) If $a = 0.1, b = 2, c = 1, d = 0.1$ then ϕ has a unique 2-cycle $C = (0.1118, 169.4132)$.
- (iii) If $a = 0.21, b = 2.1, c = -2.8, d = 1.3$ then ϕ has two 2-cycles $C_1 = (0.2593, 41.2206)$ and $C_2 = (0.3525, 13.3090)$.
- (iv) If $a = 0.18, b = 2.1, c = -2.8, d = 1.3$ then ϕ has three 2-cycles as follow

$$C_1 = (0.2001, 102.9321), C_2 = (0.4058, 7.8071), C_3 = (0.7646, 1.0453).$$

The following theorem treats the convergence of solutions of Eq.(1.1) when $c \in (c_-, c^*)$ and ϕ has a unique equilibrium \bar{t} with $x_M \leq \bar{t}$ or $x_m \leq \bar{t} \leq x_M$.

Theorem 3.2. Assume that $c \in (c_-, c^*)$, ϕ has a unique equilibrium \bar{t} , $x_M \leq \bar{t}$ or $x_m \leq \bar{t} \leq x_M$, and $\{t_n\}_{n=0}^\infty$ is a positive solution for Eq.(1.1).

- (a) either ϕ has no 2-cycle or, it has two 2-cycles $(p_1, q_1), (p_2, q_2)$ with $p_1 \leq p_2 < x_m < \bar{t} < q_2 \leq q_1$ (note that if $p_1 = p_2 = p$ then there is essentially one 2-cycle and p is an iterated root (of order 2) of the polynomial G in Lemma 3.1).
- (b) If ϕ has no 2-cycle then $\{t_n\}$ converges to \bar{t} .
- (c) Assume that ϕ has two 2-cycles $(p_1, q_1), (p_2, q_2)$ with $p_1 \leq p_2 < \bar{t} < q_2 \leq q_1$. Then $\{t_n\}$ converges to \bar{t} if $t_0 \in (p_2, q_2)$ and converges to the 2-cycle (p_2, q_2) if $t_0 = p_2$ or q_2 . Otherwise, $\{t_n\}$ converges to the 2-cycle (p_1, q_1) .

Proof. We only give the proof for the case $x_M \leq \bar{t}$. The proof of the case $x_m \leq \bar{t} \leq x_M$ is more easier and somehow similar and will be omitted. Before proceeding to proof note that ϕ is decreasing on the intervals $(0, x_m]$ and $[x_M, \infty)$, and increasing on the interval $[x_m, x_M]$. Also $\phi^2(t) \rightarrow a$ as $t \rightarrow 0^+$ and by Theorem 2.3(a), $(\phi(t) - t)(t - \bar{t}) < 0$ for all $t > 0$, $t \neq \bar{t}$. We use these facts in the proof frequently (without mentioning them again).

(a) At first we show that $\phi^2(t) > t$ on $[x_m, \bar{t})$. By Lemma 3.2(d) $\phi^2(t) > t$ on $[x_M, \bar{t})$. So it suffices to show that $\phi^2(t) > t$ on $[x_m, x_M]$. Assume that $t \in [x_m, x_M]$. Then $t < \phi(t) < \phi(x_M)$. So, either $t < \phi(t) \leq \bar{t}$ or, $\bar{t} < \phi(t) < \phi(x_M)$. In the former case Theorem 2.3(a) yields $t < \phi(t) < \phi^2(t)$ and in the later case Lemma 3.2(c) implies that $t < x_M < \phi^2(x_M) < \phi^2(t)$.

Next, (by the previous discussions) if $\phi^2(t) > t$ on $(0, x_m)$ then ϕ has no 2-cycle. Otherwise, there exists $p < x_m$ such that either ϕ^2 crosses the 45 degree line at p or, it is tangent to the 45 degree line at p . Let $q = \phi(p)$. In the former case since q is another period 2 point for ϕ and G is a polynomial of degree 6 then by the intermediate value theorem there should exists exactly another 2-cycle for ϕ other than (p, q) . In the later case p is an iterated root for G and by the same reasons we obtain that ϕ has no other 2-cycle.

(b) Note that in this case

$$(\phi^2(t) - \bar{t})(t - \bar{t}) < 0, \quad t \neq \bar{t}. \quad (3.10)$$

Consider the interval $I = [x_M, \phi(x_M)]$. Using Lemma 3.2, it's easy to verify that I is invariant and ϕ^2 is increasing on I . These facts together with (3.10) imply that $\{t_n\}$ converges to \bar{t} if $t_0 \in I$. If we show that all of iterates of ϕ will eventually end up in I then the proof is complete. Note that it's sufficient to prove this claim when $t_0 \in [x_m, x_M) \cup (\phi(x_M), \infty)$. Now, suppose that $t_0 \in [x_m, x_M)$. Then $t_n \in (x_m, \phi(x_M))$ for all $n \in \mathbb{N}$. If $t_n \in (x_m, x_M)$ for all $n \in \mathbb{N}$ then by Theorem (2.3)(a), $\{x_n\}$ is increasing and hence convergent to a number in (x_m, x_M) which simply is a contradiction. Thus $\{t_n\}$ eventually ends up in I in this case.

On the other hand suppose that $t_0 \in (\phi(x_M), \infty)$. We claim that for some $n_0 \in \mathbb{N}$ $t_{n_0} \in [x_m, \phi(x_M)]$ and therefore we are done. For the sake of contradiction, assume that such a claim is not true. Then, $t_{2n} > \phi(x_M)$ and $t_{2n+1} < x_m$ for all $n \in \mathbb{N}$. So by (3.10) it could be shown that $\{t_{2n}\}$ is decreasing and $\{t_{2n+1}\}$ is increasing. Hence, $\{x_n\}$ converges to a 2-cycle, a contradiction.

(c) In this case we have

$$(\phi^2(t) - t)(t - p_1)(t - p_2)(t - \bar{t})(t - q_2)(t - q_1) < 0, \quad t \neq p_1, p_2, \bar{t}, q_1, q_2.$$

By (a) we know that $p_1 < p_2 < x_m$. Using this fact and Theorem 2.3(a) it's easy to show that ϕ^2 is increasing if $t \in (0, p_2) \cup (q_2, \infty)$. Therefore, if $t_0 \in (0, p_2) \cup (q_2, \infty)$ then the proof is similar to Theorem 3.1 and will be omitted. On the other hand, if $t_0 \in (p_2, q_2)$ then the proof is exactly similar to what is applied in (b). The proof is complete.

Remark 3.2. The following examples are some examples for all of cases in Theorem 3.2. Examples (i) and (ii) represent cases (b) and (c) (respectively) when $x_M \leq \bar{t}$ while examples (iii) and (iv) represent the same cases (respectively) when $x_m \leq \bar{t} \leq x_M$.

- (i) If $a = 1, b = 5, c = -4, d = 1$ then ϕ has no 2-cycle.
- (ii) If $a = 0.1, b = 5, c = -4, d = 1$ then ϕ has two 2-cycles $C_1 = (0.1111, 450.5876)$ and $C_2 = (0.2019, 48.2751)$.
- (iii) If $a = 0.15, b = 4, c = -4, d = 1.1$ then ϕ has no 2-cycle.
- (iv) If $a = 0.1, b = 4, c = -4, d = 1.1$ then ϕ has two 2-cycles $C_1 = (0.1068, 590.5885)$ and $C_2 = (0.2378, 28.0116)$.

The following theorem discusses (in some details) about the convergence of solutions of Eq.(1.1) when $c \in (c_-, c^*)$ and ϕ has a unique equilibrium \bar{t} with $\bar{t} < x_m$.

Theorem 3.3. Assume that $c \in (c_-, c^*)$, ϕ has a unique equilibrium \bar{t} with $\bar{t} < x_m$, and the sequence $\{t_n\}_{n=0}^\infty$ is a positive solution for Eq.(1.1). Consider the quantities c_1^* and η in Lemma 3 and let $I = [\phi(x_m), \phi^2(x_m)]$.

- (a) Under the following hypothesis the interval I is invariant.
 - (H) Either $c \leq c_1^*$ or, $c > c_1^*$ but $\phi^2(x_m) \leq \eta$.
Moreover, if $c \leq c_1^*$ then all iterations of ϕ will eventually end up in I .
- (b) If ϕ has no 2-cycle then $\{t_n\}$ converges to \bar{t} .
- (c) Assume that ϕ has one 2-cycle (p, q) with $p < \bar{t} < q \leq x_m$. Let $\mathcal{S} = \{\phi^{-n}(\bar{t})\}_{n=0}^\infty$. If $\bar{t} < -dx_M/(cx_M + 2d)$ then $\mathcal{S} = \{\bar{t}\}$. Also, if $t_0 \notin \mathcal{S}$ then $\{t_n\}$ converges to the 2-cycle (p, q) otherwise, it converges to \bar{t} .

Proof. Before proceeding to proof note that Theorem (2.3)(a), monotonic properties of ϕ , and this fact that a is a horizontal asymptote of ϕ are used in the proof frequently. So we don't mention them again.

(a) Suppose that $t \in I$. We show that $\phi(t) \in I$ if (H) holds. At first assume that $c \leq c_1^*$. So by Lemma 3.3(b) $\phi(x_m) \leq a$. Therefore, $\phi(t) > \phi(x_m)$ for all $t > 0$. It remains to show that $\phi(t) \leq \phi^2(x_m)$. It's easy to verify that $\phi(t) \leq \max\{\phi^2(x_m), x_M\}$. If $\max\{\phi^2(x_m), x_M\} = \phi^2(x_m)$ then we are done. Thus, assume that $\phi^2(x_m) < x_M$. If $\phi^2(x_m) \leq x_m$ or $\phi^2(x_m) > x_m$ but $t < x_m$ then $t < x_m$ and hence $\phi(t) < \phi^2(x_m)$. Thus, suppose that $x_m < t \leq \phi^2(x_m)$. Therefore $\phi(t) < \phi^3(x_m) < \phi^2(x_m)$. Hence $\phi(t) \in I$.

Next assume that $c > c_1^*$ and $\phi^2(x_m) \leq \eta$. Similar to the previous discussions it could be shown that $\phi(t) \leq \phi^2(x_m)$. It remains to show that $\phi(x_m) \leq \phi(t)$. This matter is obvious If $t \leq x_M$. Thus, suppose that $t > x_M$. Therefore, since $t \leq \phi^2(x_m) \leq \eta$

$$\phi(t) \geq \phi^3(x_m) \geq \phi(\eta) = \phi(x_m),$$

where the equality $\phi(\eta) = \phi(x_m)$ holds by the definition of η in Lemma 3.3. This proves the invariance of I .

Finally assume that $c \leq c_1^*$. By Lemma 3.3(b) $\phi(x_m) \leq a$. So $t_n \geq \phi(x_m)$ for all $n \geq 1$. As a result, if there exists $n_0 \in \mathbb{N}$ such that $t_{n_0} \leq \phi^2(x_m)$ then $t_{n_0} \in I$ and we are done. Thus we assume that $t_n > \phi^2(x_m)$ for every $n \geq 1$. Consequently, the sequence $\{t_n\}$ is decreasing and hence convergent to a number greater than or equal to $\phi^2(x_m)$, a contradiction. Therefore, all iterates of ϕ will eventually end up in I .

(b) Note that in this case (3.10) holds and the proof is somehow similar to Theorem 3.2(b) and therefore it will be omitted.

(c) Note that if $\phi(x_M) < \bar{t}$ then $\mathcal{S} = \{\bar{t}\}$ obviously. Using (3.2) and some algebra somehow similar to what is applied in Lemma 3(c) one can write

$$\phi(x_M) - \bar{t} = -\frac{(x_M - \bar{t})^2}{x_M^3 \bar{t}^3} [(cx_M + 2d)\bar{t} + dx_M],$$

so $\phi(x_M) < \bar{t}$ if and only if $(cx_M + 2d)\bar{t} + dx_M > 0$. By (3.6) the later inequality is equivalent to $\bar{t} < -dx_M/(cx_M + 2d)$. Therefore, if $\bar{t} < -dx_M/(cx_M + 2d)$ then $\mathcal{S} = \{\bar{t}\}$.

On the other hand, it's easy to show that in this case the following inequality holds

$$(\phi^2(t) - t)(t - p)(t - \bar{t})(t - q) < 0, \quad t \neq p, \bar{t}, q. \quad (3.11)$$

With the help of (3.11) and an analysis somehow similar to the previous theorems the rest of proof is easy and will be omitted. The proof is complete.

Remark 3.3. *In both of the following examples hypothesis (H) in Theorem 3.3 holds. Also, (i) and (ii) represent cases (b) and (c) in Theorem 3.3 respectively.*

(i) *If $a = 0.7, b = 2.2, c = -3, d = 1$ then ϕ has no 2-cycle.*

(ii) *If $a = b = 1, c = -3.3, d = 3$ then ϕ has a unique 2-cycle $C = (1.1687, 1.3190)$.*

The following theorem discusses (in some details) about the convergence of solutions of Eq.(1.1) when ϕ has two equilibria. Theorem 2 together with Lemma 2 play an essential role for it's proof. But, it's proof will be omitted since it is somehow similar to the proofs of some of the previous theorems of this section. Also, a similar theorem exists when ϕ has three equilibria. So it will be omitted.

Theorem 3.4. *Assume that $c \in (c_-, c^*)$, ϕ has two equilibria \bar{t}_1, \bar{t}_2 with $\bar{t}_1 < \bar{t}_2$, and $\{t_n\}_{n=0}^\infty$ is a positive solution for Eq.(1.1). Also consider the values c_m and c_M of the parameter c in Theorem 2.2. Then there are two cases as follow:*

(a) *$c = c_m$; assume that $\delta < x_m$ is the (unique) number such that $\phi(\delta) = \bar{t}_2$ and $\bar{t}_2 \leq a$. In this case the following cases are possible:*

(a₁) *$x_m \leq \bar{t}_1$; If $t_0 \in (\delta, \bar{t}_2)$ then $\{t_n\}$ converges to \bar{t}_1 otherwise, it converges to \bar{t}_2 .*

(a₂) $\bar{t}_1 < x_m$; In this case we consider the following cases:

(a₂₁) ϕ has no 2-cycle; If $t_0 \in (\delta, \bar{t}_2)$ then $\{t_n\}$ converges to \bar{t}_1 otherwise, it converges to \bar{t}_2 .

(a₂₂) ϕ has one 2-cycle (p, q) with $p < \bar{t}_1 < q \leq x_m$; Let $I = [\phi(x_m), \phi^2(x_m)]$ and $\mathcal{S} = \{\phi^{-n}(\bar{t}_1)\}_{n=0}^{\infty}$. Then, $\{t_n\}$ converges to the 2-cycle (p, q) if $t_0 \in (\delta, \bar{t}_2) \setminus \mathcal{S}$ and converges to \bar{t}_1 if $t_0 \in \mathcal{S}$. Otherwise, it converges to \bar{t}_2 . In particular, if $t_0 \in I \setminus \{\bar{t}_1\}$ then $\{t_n\}$ converges to the 2-cycle (p, q) .

(b) $c = c_M$; assume that $\delta' < x_m$ is the (unique) number such that $\phi(\delta') = \bar{t}_1$ and $\bar{t}_1 \leq a$. Then $\{t_n\}$ converges to \bar{t}_1 if $t_0 \in [\delta', \bar{t}_1]$ otherwise, it converges to \bar{t}_2 .

Remark 3.4. In Theorem 3.4(a) and Theorem 3.4(b) it is assumed, for the sake of simplicity, that $\bar{t}_2 \leq a$ and $\bar{t}_1 \leq a$ respectively. These assumptions are not necessary. Since similar theorems exist without these assumptions we don't mention them. Also, the following examples represent cases (a₁), (a₂₁), (a₂₂), and (b) in Theorem 7, respectively.

(i) $a = 1, b = 2.4, c = -3.8, d = 1.4$.

(ii) $a = 1, b = 2, c = -3, d = 1$.

(iii) $a = 1, b = 1.9, c = -2.8, d = .9$ with one 2-cycle $C = (0.5573, 0.5937)$.

(iv) $a = 2, b = .5, c = -3, d = 1.5$.

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